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BRILLIANT POINTS OF A FAMILY OF CONCENTRIC SPHERES.

BY ALSTON HAMILTON.

A brilliant point on the surface of a sphere is a point at which a ray of light, emanating from a fixed source, will be reflected to meet the eye of an observer. The problem is to construct the locus of such points for a family of concentric spheres.

Assume a point, O, in space as the centre of the spheres. Assume arbitrarily the points E and D, also in space, either of which may be the source of light and the other the point of vision. Through O, E, and D pass a plane. This plane will contain the brilliant point of every sphere for the given points D and E.

Assume OX and OY at right angles to each other. Use these as axes of X and Y respectively. Let the coordinates of E be (a, b) and those of D, (c, d). Assume a circle whose radius is r and centre, O. The equation of this circle is $x^2 + y^2 = r^2$. This circle is the one cut from one of the spheres by the XY plane. Select a point A on the circumference of this circle and draw AD and AE. Draw the normal and the tangent at A. Let the coordinates of A be (x', y'). Now, in order that A shall be the brilliant point for this sphere EAC must be equal to CAD; expressing this relation analytically we are led to the equation:

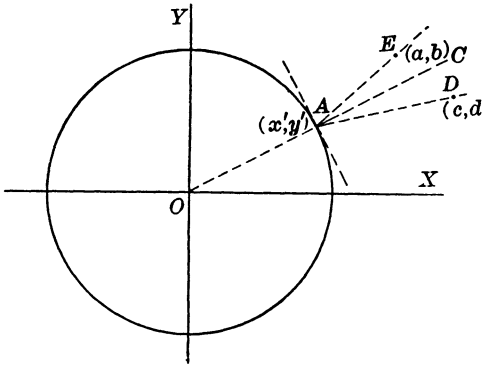

FIG. 1.

$$(b + d)(x^3 + xy^2) - (a + c)(y^3 + yx^2) + 2(ac - bd)xy + (ad + bc)(y^2 - x^2) = 0 \qquad (1)$$

where x' and y' have been replaced by x and y.

This is the equation satisfied by the coordinates of the brilliant points and also by the coordinates of the so-called "virtual brilliant points." A is a *virtual brilliant point* if the tangent at A to the sphere bisects the angle EAD.

The equation is perfectly general, but unsatisfactory for tracing the curve. The following method has been employed to simplify the process while detracting nothing from its applicability to any case.

Let the coordinate axes be turned through an angle α such that in the new system the coefficient of $(y^2 - x^2)$ will disappear. α is given by the equation

$$\tan 2\alpha = \frac{ad + bc}{ac - bd} \cdot \tag{2}$$

The angle 2α is thus seen to be equal to the sum of the angles that the right lines drawn through the origin to the points D and E make with the axis of X. Equation (1) becomes

$$x^3 + xy^2 - M(x^2y + y^3) + Qxy = 0. \tag{3}$$

Let (A, B) and (C, D) respectively be the coordinates of E and D referred to the new axes. Then

$$\left.\begin{array}{ll} A = a \cos \alpha + b \sin \alpha, & C = c \cos \alpha + d \sin \alpha, \\ B = b \cos \alpha - a \sin \alpha, & D = d \cos \alpha - c \sin \alpha, \\ M = \dfrac{A + C}{B + D}, & Q = \dfrac{2(AC - BD)}{B + D} \cdot \end{array}\right\} \tag{4}$$

Now transform the equation (3) to polar coordinates having the new axis of X (or X') as initial line. The equation then becomes

$$r = \frac{Q \sin \theta}{M \tan \theta - 1} \cdot \tag{5}$$

To trace the curve by means of (5) it is necessary to consider only values of θ between 0 and π, since an addition of π to θ changes the sign of r and hence gives no new point of the curve. From (5)

$$\frac{dr}{d\theta} = \frac{Q(M \sin \theta - \cos \theta - M \sin \theta \sec^2 \theta)}{(M \tan \theta - 1)^2} \cdot \tag{6}$$

Place $u = 1/r$; then

$$\frac{du}{d\theta} = \frac{M \sin \theta}{Q \cos^2 \theta} + \frac{\cos \theta}{Q \sin^2 \theta}$$

and

$$u + \frac{d^2u}{d\theta^2} = \frac{2\,M}{Q \cos^3 \theta} - \frac{2}{Q \sin^3 \theta} \cdot \tag{7}$$

For a maximum radius vector

$$\frac{dr}{d\theta}=0\,;\quad\text{whence, from (6),}\quad \tan\theta=\frac{-1}{\sqrt[3]{M}}\,;\qquad r=\frac{-Q}{(1+M^{\frac{2}{3}})^{3/2}}\,.\qquad(8)$$

For the asymptotes place

$$\infty=r=\frac{Q\sin\theta}{M\tan\theta-1}\,;\quad\text{whence}\quad\tan\theta=\frac{1}{M}\,.\qquad(9)$$

But

$$\frac{r^2d\theta}{dr}=\frac{-Q\sin\theta}{M\tan^2\theta+\cot\theta}=\frac{-MQ}{(M^2+1)^{3/2}},$$

which is finite and represents the distance of the origin from the asymptote. Hence there is always an asymptote. To draw it, draw a line through the origin making an angle with the initial line $\tan^{-1}1/M$, and draw another line through the origin at right angles to this. On this second line, lay off the distance $r^2d\theta/dr$ to the right (looking from the origin in the direction of the line first drawn) if $r^2d\theta/dr$ is positive, or to the left if $r^2d\theta/dr$ is negative. Through the extremity of this line, draw a line parallel to the line first drawn. It will be an asymptote.

For a point of inflexion,

$$u+\frac{d^2u}{d\theta^2}=0,$$

and from (7),

$$\tan\theta=\frac{1}{\sqrt[3]{M}},\qquad r=\frac{Q}{(M^{\frac{2}{3}}-1)\sqrt{M^{\frac{2}{3}}+1}}\,.$$

When ϕ is the angle made by the tangent at the point of inflection with the radius vector

$$\tan\phi=\frac{rd\theta}{dr}=-\frac{1}{2}(M^{\frac{1}{3}}-M^{-\frac{1}{3}}).$$

For a multiple point,

$$f=0,\qquad\frac{\partial f}{\partial x}=0,\qquad\frac{\partial f}{\partial y}=0,$$

where $f(x,y)$ represents the first member of equation (3). These equations are satisfied only at the origin, and the two tangents at this point are evidently $x=0$, $y=0$ corresponding to the values $\theta=0$ and $\frac{\pi}{2}$.

Example. Let D be $(-6,\ 2)$ and E be $(4,\ 7)$ referred to the first system of coordinates. Then $2a=\tan^{-1}\frac{17}{19}$, $A=5.1$, $B=-6.25$, $C=4$,

$D = 4.9$, $M = -6.75$, $Q = -75.6$. From (5), $r = \dfrac{75.6 \sin \theta}{6.75 \tan \theta + 1}$. For a maximum radius vector $\tan \theta = .53$, $r = 7.7$. For the asymptote, $\tan \theta = -.15$, $r^2\, d\theta/dr = 1.61$. For the point of inflexion $\tan \theta = -.526$, $r = 13.8$, $\tan \phi = .69$

The above points having been determined, the following geometric method will save labor in determining other points. From each of the points *D* and *E* draw two tangents to the circle *BAC* arbitrarily assumed with centre at *O*, but not including *D* or *E*. These tangents will be *ER*, *EB*, *DT*, *DA* and will intersect in four points *F*, *G*, *H*, *K*. The four points thus found will be real or virtual brilliant points, as is at once evident from a glance at the figure (Fig. 2). By varying the circle *BAC* any number of such points may be found. Draw *EO* and *DO* and prolong them. The angles *EOW* and *DOS* include all *virtual* brilliant points and the angles *EOD* and *WOS* include all *real* brilliant points.

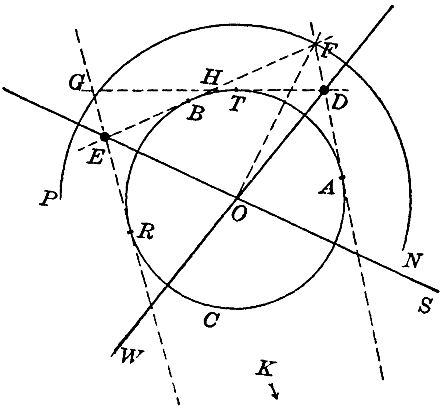

FIG. 2.

Figure 3 shows the description by curve-tracing methods of the example given. It is to be noticed that the asymptote is parallel to the tangents drawn from the eye and light respectively to the circle for which the tangents will be parallel to each other. The direction of the asymptote may thus be checked graphically. The above is readily seen from the description of the curve by points by the graphical method.

Figure 4 shows the description of the same curve by the graphical method. In both figures the part of the curve containing virtual brilliant points is dotted. A combination of the two methods, using the graphical method for intermediate points gives more satisfactory results than either alone.*

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* Extract from Lieutenant Hamilton's letter to the Editor: "The problem as discussed would seem to indicate that use may be made of its law, analytically expressed, in constructing *range-finders*. Since solving it I have, indeed, planned from it several varieties of range-finder; but with a small instrument the base is too small to give very accurate results, and difficulties in the way of sufficiently accurate construction abound to such an extent as to disappoint me with the result."

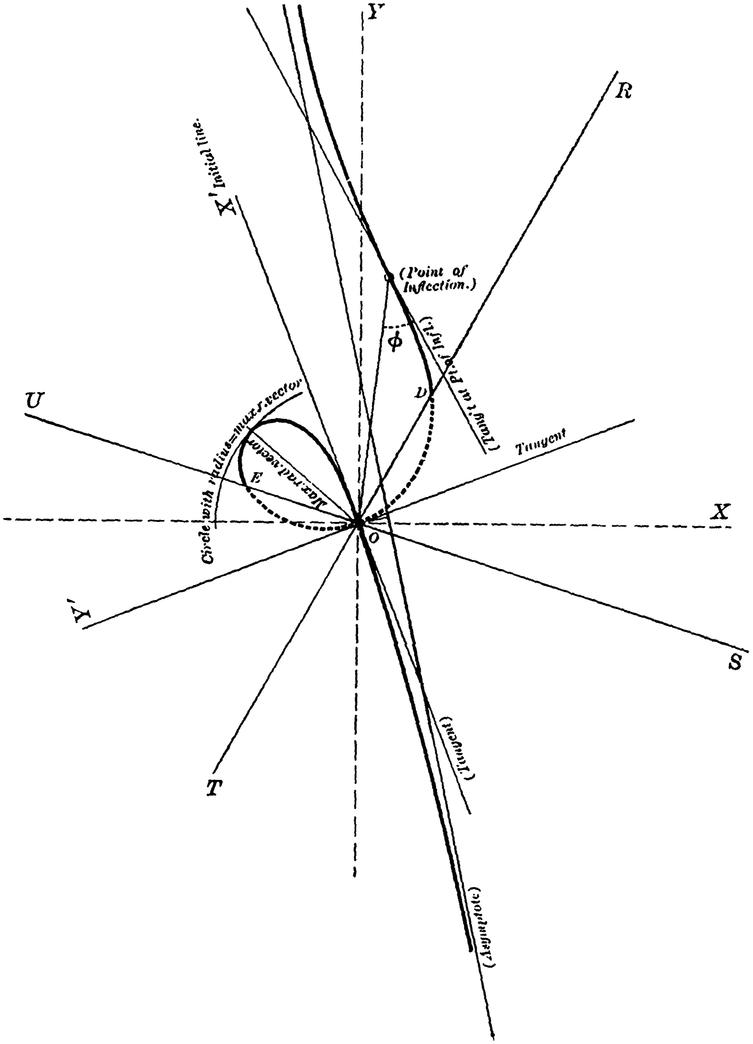
Y
R
X' Initial line.
(Point of Inflection.)
φ
(Tang't at Pt. of Inf.)
D
U
Circle with radius=max.r.vector
Max.rad.vector
E
Tangent
X
O
Y'
S
T
(Tangent)
(Asymptote)

FIG. 3.

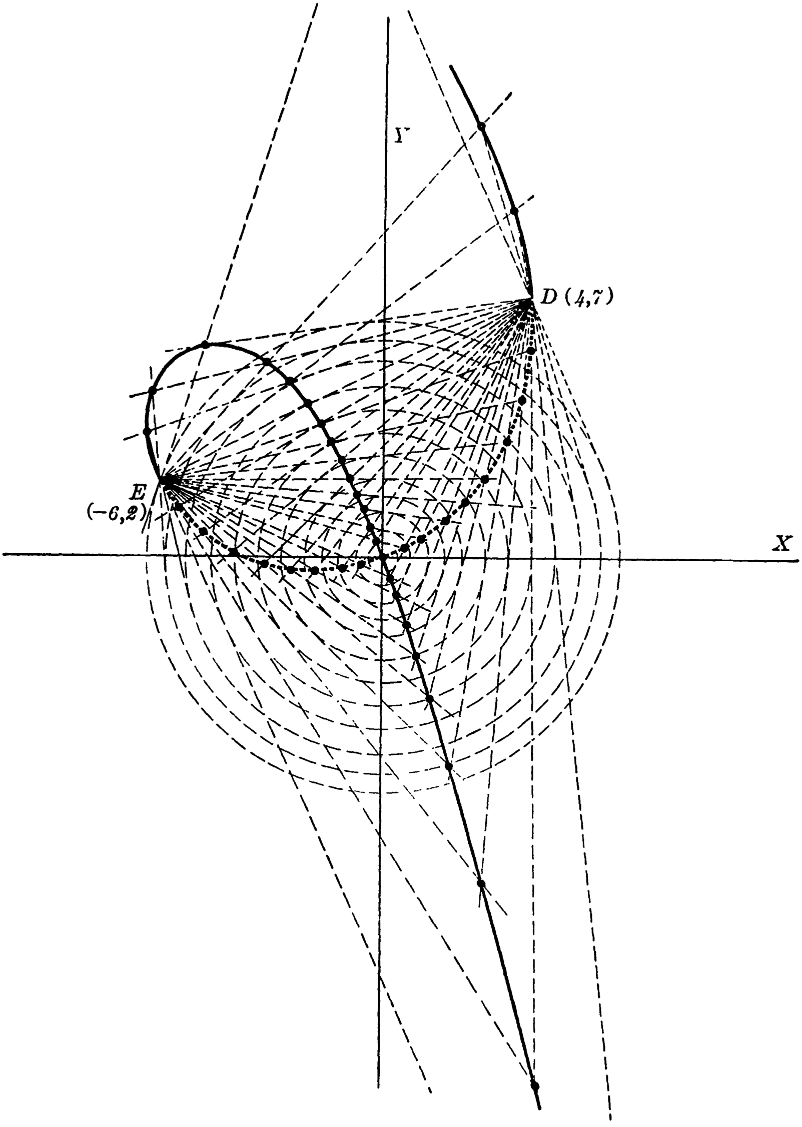

FIG. 4.

ANGELES, LUZON, PHILIPPINE ISLANDS, APRIL 12, 1900.